A convex view of query (and control) algorithms

Duyal Yolcu

February 20, 2024

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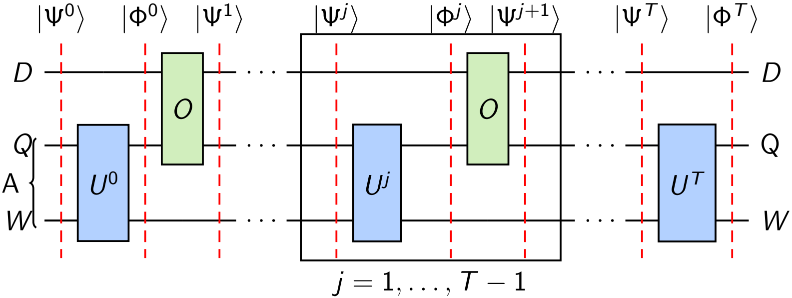
- We will discuss an alternative, where we keep track of *collections of memory states, modulo equivalences*.
- Formally (pure-state quantum): The matrix of inner products (*Gram matrix*) of wavefunctions for different possible inputs.

The general view: Control



- An **agent** interacts with its **environment** in discrete time

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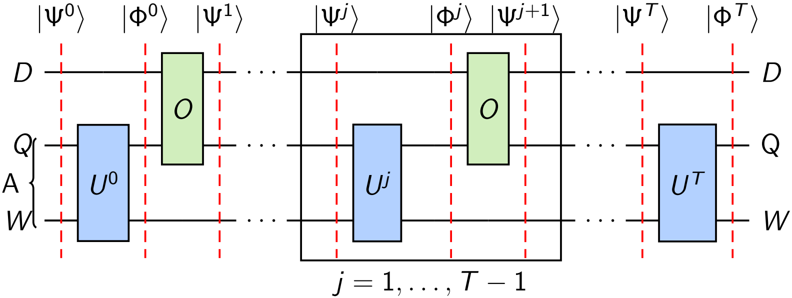

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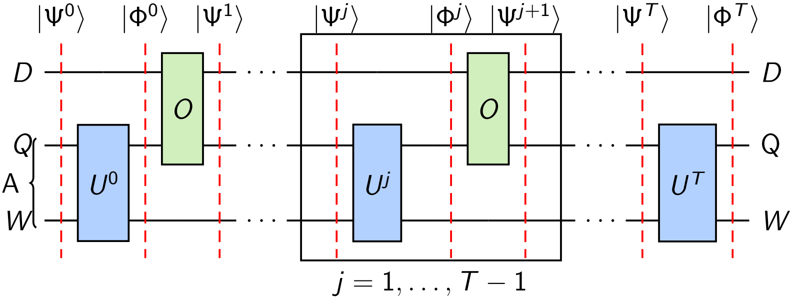

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- $|\Psi^0\rangle$ and $|\Phi^T\rangle$ contain information about both state of environment and knowledge about environment

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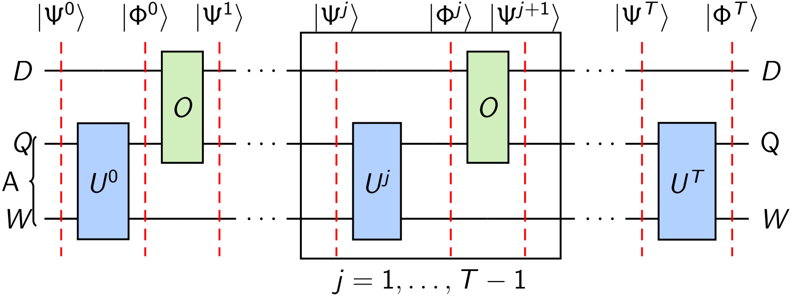

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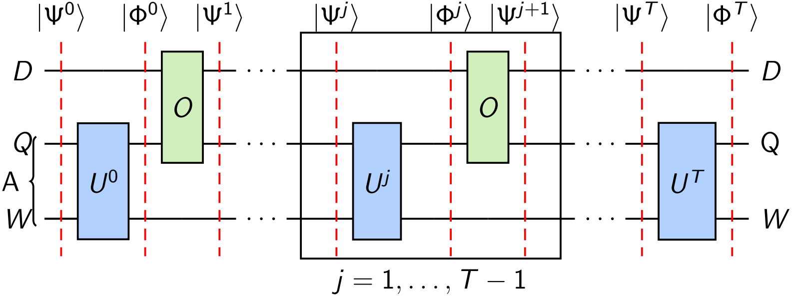

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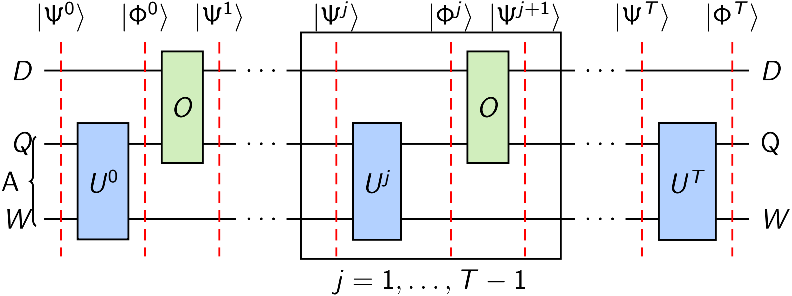

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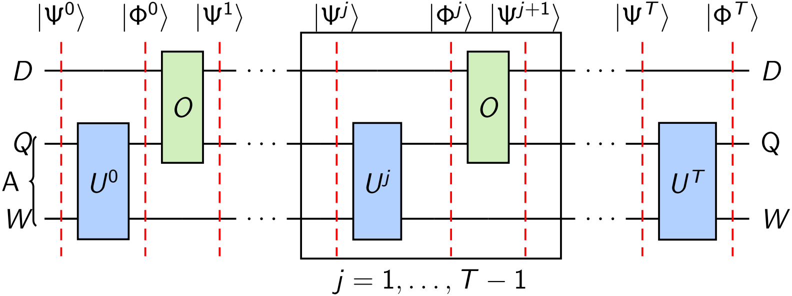

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- mixed-state quantum: completely positive trace-nonincreasing

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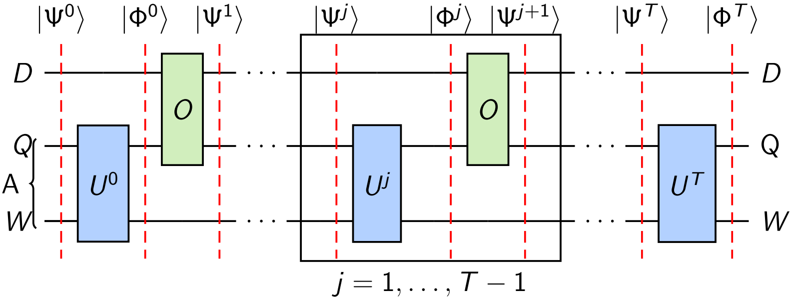

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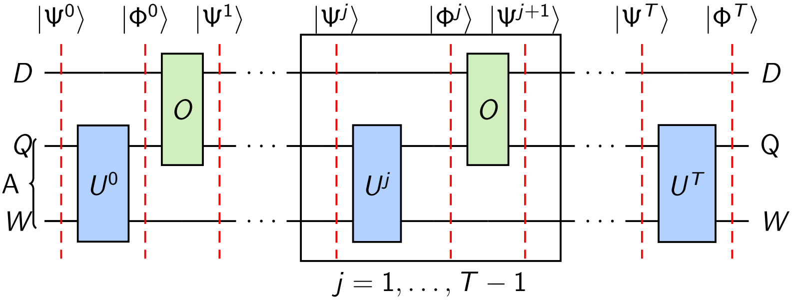

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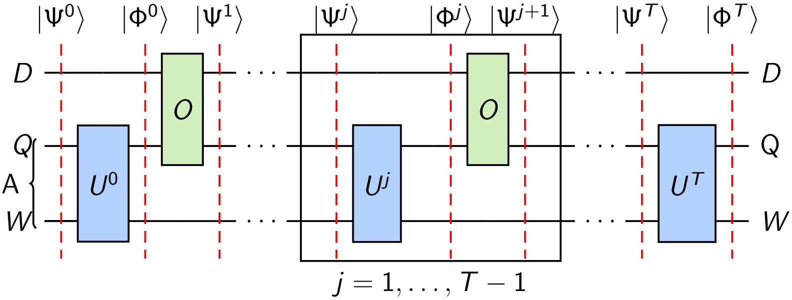

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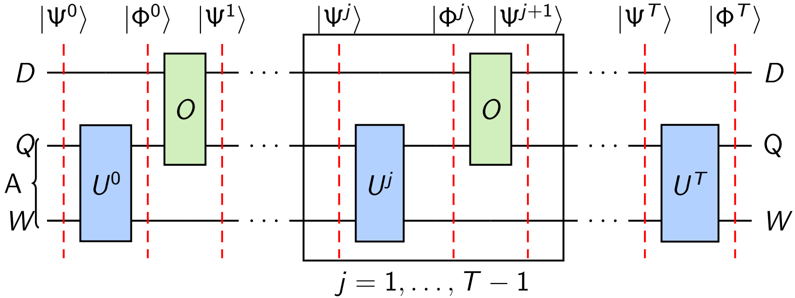

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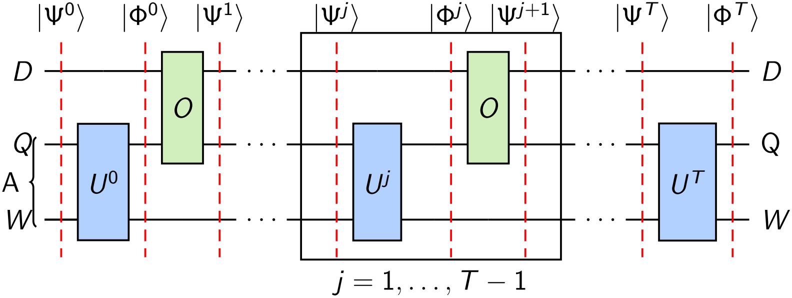
Control problems: Examples



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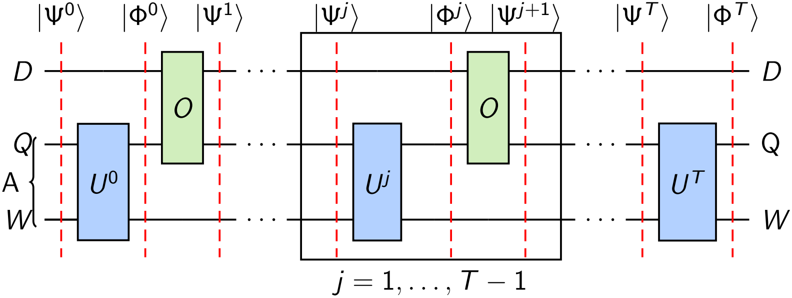

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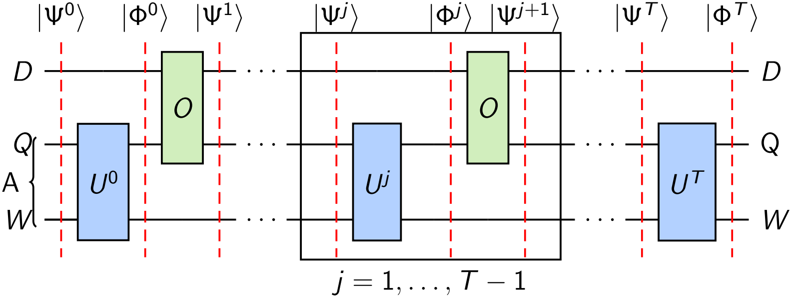

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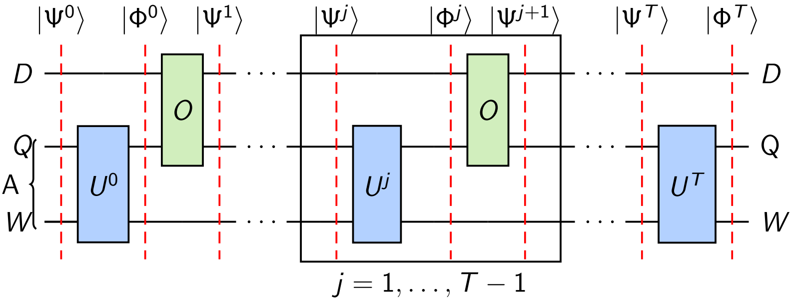

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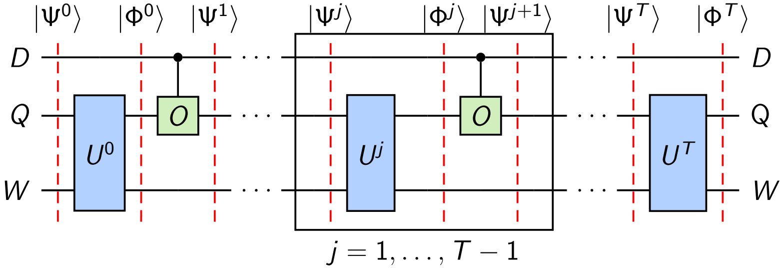
- Robot cooks from fridge

Special case: Query problems



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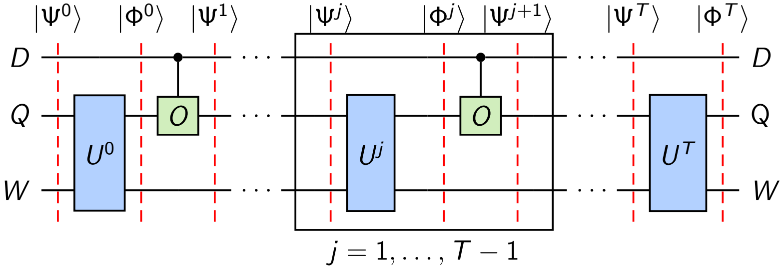

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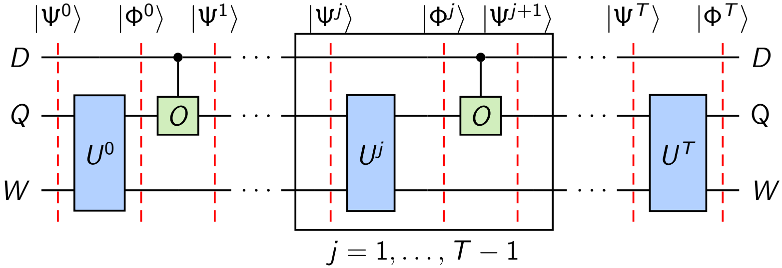

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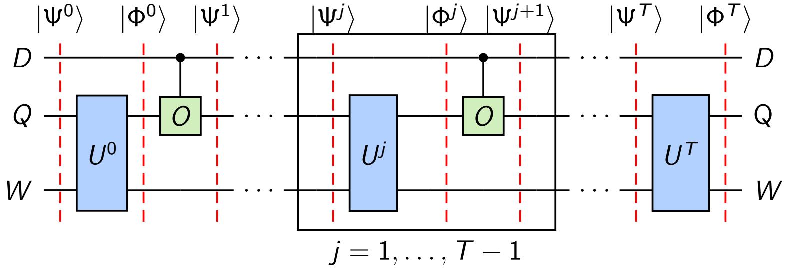

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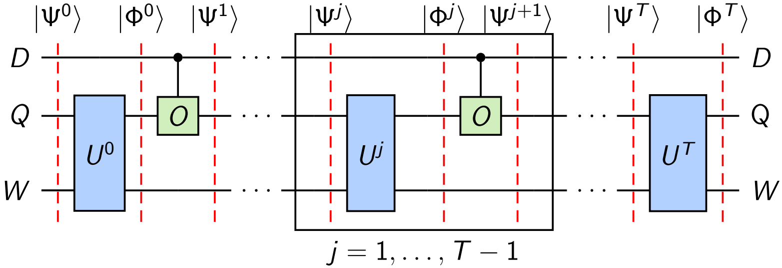

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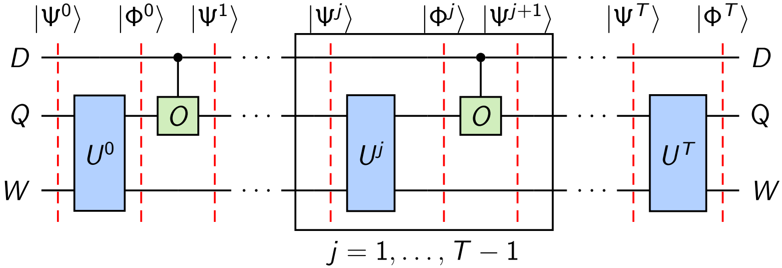

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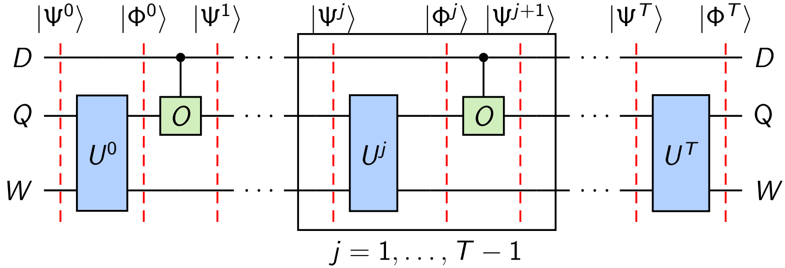

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- Classical: Active learning/Optimal experimental design

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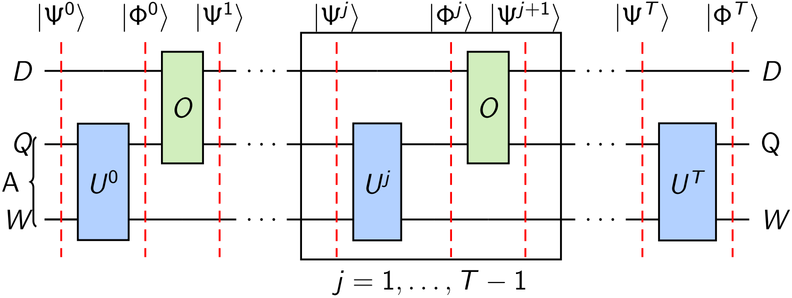
- The **adversary bound** is a lower-bound method generalizing the unstructured search lower bound to other problems.

States, state collections, states of knowledge



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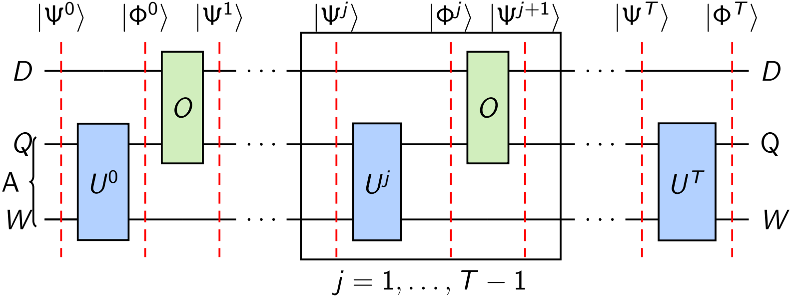

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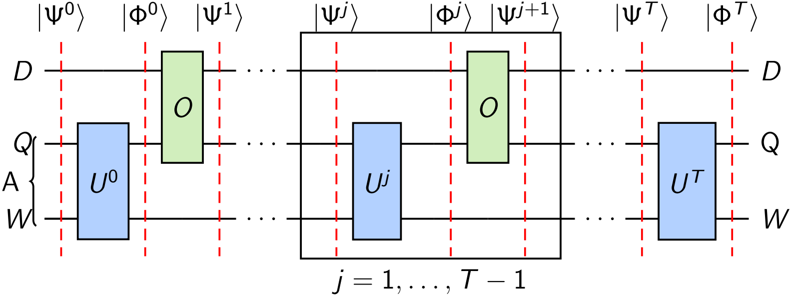

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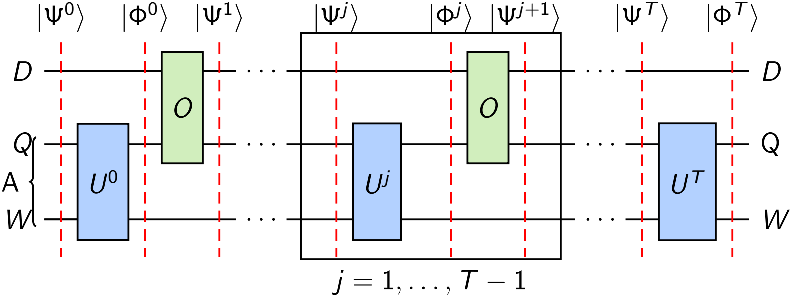

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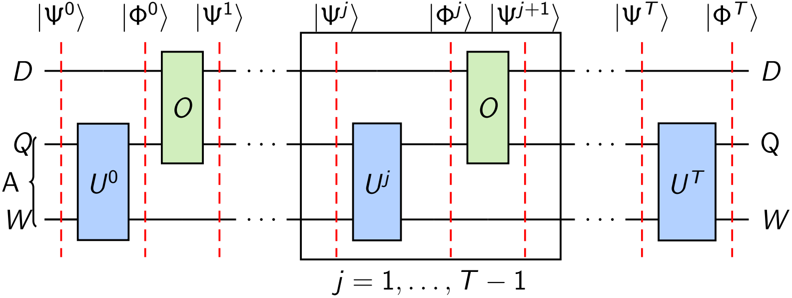

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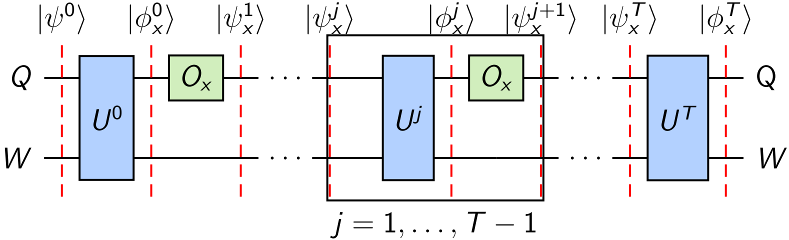
- For Gram matrices (pure-state quantum): Regular convex combinations
- If $G_1 \in \mathcal{S}^{+D}$ is reachable from $G_0 \in \mathcal{S}^{+D}$ in k queries, and $G'_1 \in \mathcal{S}^{+D}$ is reachable from G'_0 in k queries, then $pG_1 + p'G'_1$ is reachable from $pG_0 + p'G'_0$ in k queries as well by conditional execution

State conversion and the Gram matrix picture



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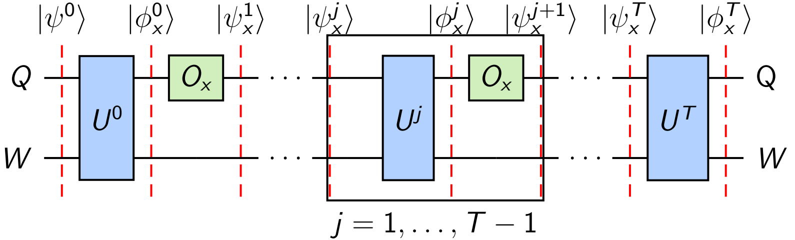

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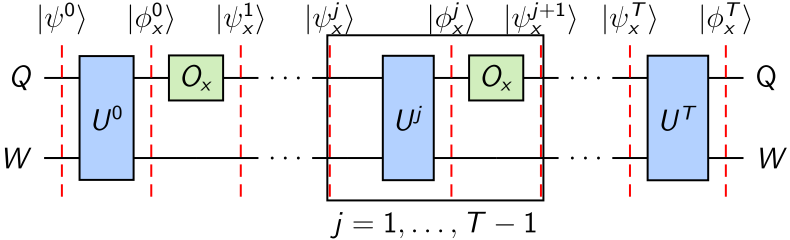

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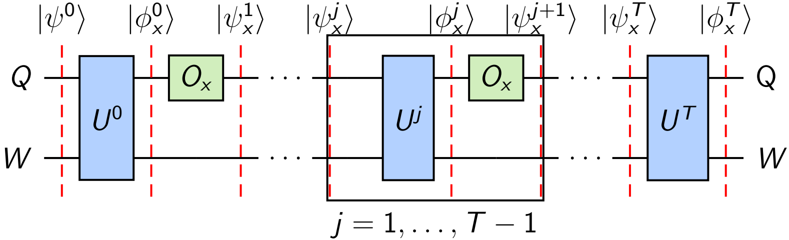

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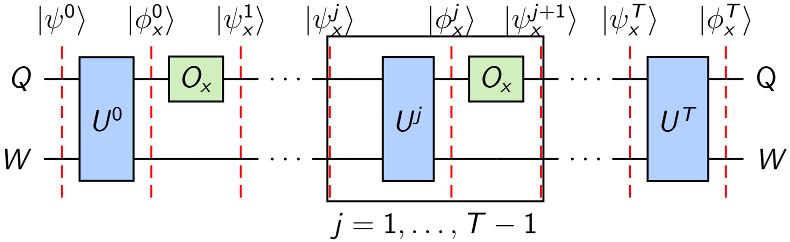
State conversion and the Gram matrix picture



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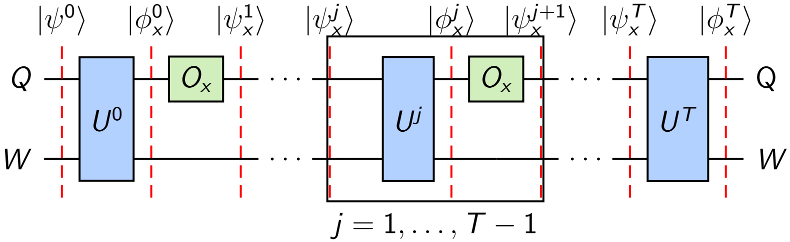

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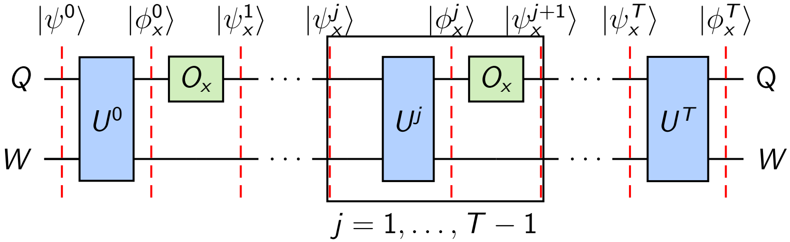

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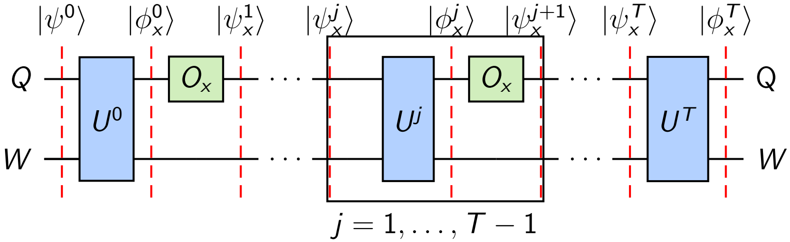

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- So the following is a lower bound on the number of queries needed to transform G_0 to G_T:

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- For classical probability theory/mixed-state quantum: Not so easy, because space is infinite-dimensional

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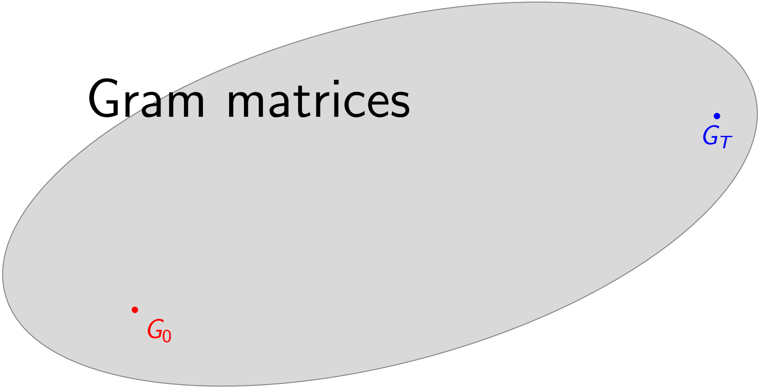
- As $T' \to \infty$, initial and final states converge to G_0 or G_T.
$\to$ Approximate solution of state conversion problem $G_0 \to G_T$

Picture



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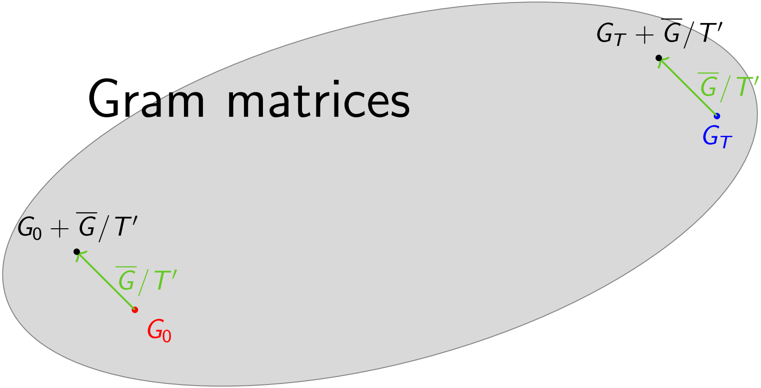

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$G_T + \overline{G}/T'$
$\overline{G}/T'$
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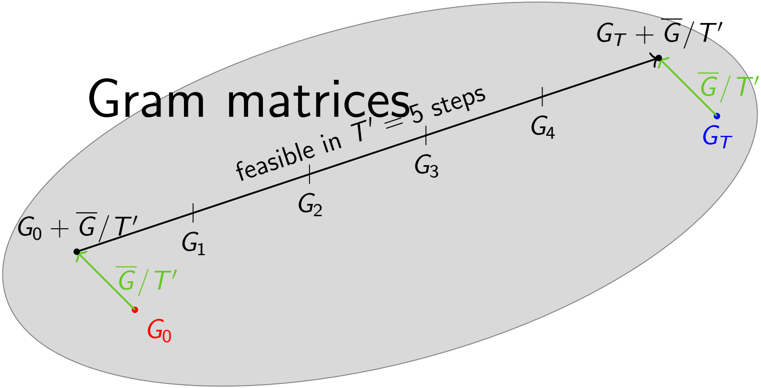

Gram matrices
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G_4
G_3
G_2
G_1
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- Any ideas? Let's have another call :)
- fi1w+qudent@outlook.com
- Thanks to Alexander Belov

Bonus: Algebra of SOKs/Formalizing reachability

$$\left[(|\psi_{x,q}\rangle)_{x\in D}\right]_{\sim} + \left[(|\psi'_x\rangle)_{x\in D}\right]_{\sim} := \left[(|\psi_x\rangle \oplus |\psi'_x\rangle)_{x\in D}\right]_{\sim}$$
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- Well-defined, commutative, associative and distributive on the equivalence classes.
- On Gram matrices: $+*$ are entrywise sums/products
- Suppose our oracle O_x just emits a new state in each query, i.e. $O_x = |\theta_x\rangle$, maps $|\psi_x\rangle \to |\theta_x\rangle \otimes |\psi_x\rangle$
- This corresponds to transforming SOKs/Gram matrices $G_\psi \to G_\psi * G_\theta$

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- For control theory: need to generalize partial trace of Gram matrices

References

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- https://arxiv.org/abs/2212.04606 formalizes the algebra of knowledge in detail (draft)
- https://arxiv.org/abs/2211.16293 discusses the universal algorithm for control theory in the Gram matrix picture (equivalent to reduced density matrices)
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